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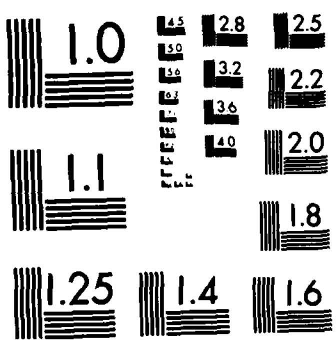

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Georgia Institute of Technology

ON THE STEP SIZE IN KARMARKAR'S ALGORITHM

by

C. M. Shetty
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PRODUCTION and DISTRIBUTION RESEARCH CENTER

SCHOOL OF INDUSTRIAL AND SYSTEMS ENGINEERING
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PDRC Report Series 85-02
March 1985

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The ~~recently~~ ~~pro~~posed algorithm of Karmarkar [1] for solving linear programs ~~uses the~~ following step: suppose $a = (a_1,\ldots,a_n) > 0$ is an interior point to the feasible region of ~~the~~ a following linear program P:

$$
\begin{array}{lll}
\text{P:} & \text{minimize} & c^t x \\
& \text{subject to} & Ax = 0 \\
& & e^t x = 1 \\
& & x > 0
\end{array}
$$

A projective transformation maps the point a to the center of the simplex a_0 and leads to ~~the following~~ a nonlinear program:

$$
\begin{array}{lll}
\text{TP:} & \text{minimize} & f(x') = \dfrac{c^t Dx'}{e^t Dx'} \\
& \text{subject to} & ADx' = 0 \\
& & e^t x' = 1 \\
& & x' > 0
\end{array}
$$

where D is a diagonal matrix with $d_{ii} = a_i$. TP is approximated by the following program:

$$
\begin{array}{lll}
\text{AP:} & \text{minimize} & c^t Dx' \\
& \text{subject to} & ADx' = 0 \\
& & e^t x' = 1 \\
& & x' > 0 \\
& & \|x' - a_0\| < \alpha r
\end{array}
$$

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where $r = [n(n-1)]^{-1/2}$, $a_0 = (1/n,\dots,1/n)$ and $0 < \alpha < 1$

The trivial solution to AP is $a_0 - \alpha r\hat{c}_p$ where $\hat{c}_p = \frac{c_p}{\|c_p\|}$ where c_p is the vector obtained by the orthogonal projection of Dc onto the region $\{ADx' = 0,\ e^t x' = 1\}$. Thus we move from the center of the sphere $B(a_0, \alpha r)$ to the boundary in the direction of $-\hat{c}_p$.

Current literature suggests use of small values of α(say, $\frac{1}{4}$) to solve the problem for mainly three reasons:

1. $c^t Dx'$ in (AP) approximates the objective function of TP better for smaller values of α.

2. The choice of $\alpha = \frac{1}{4}$, for example, allows the complexity argument to go through without adversely affecting the order of complexity.

3. Problems associated with round-off errors.

Now consider the bound on the decrease of the "potential function" derived in [1,theorem 4]

$$f(a_0) - f(a_0 - \alpha r\hat{c}_p) > \delta(n)$$

$$\text{where } \delta(n) = \alpha - \frac{\alpha^2}{2} - \frac{\alpha^2 n}{(n-1)\left[1-\alpha\sqrt{\frac{n}{n-1}}\right]} \qquad (1)$$

$$\delta = \lim_{n\to\infty} \delta(n) = \alpha - \frac{\alpha^2}{2} - \frac{\alpha^2}{1-\alpha}$$

Note that δ_{max} occurs at $\alpha^* \simeq 0.2451$. Further, for $\alpha = 1/4$ we have $\delta > 1/8$ so that $f(a_0) - f(a_0 - \alpha r\hat{c}_p) > 1/8$. However one might note that $f(x') = \frac{c^t Dx'}{e^t Dx'}$ is quasiconcave over the positive orthant and the feasible region of (TP) is convex and compact. Hence, the optimal solution to (TP) is a boundary point. Consequently it seems logical that we should solve AP for the maximum possible value of α gaining a greater reduction of the objective function.

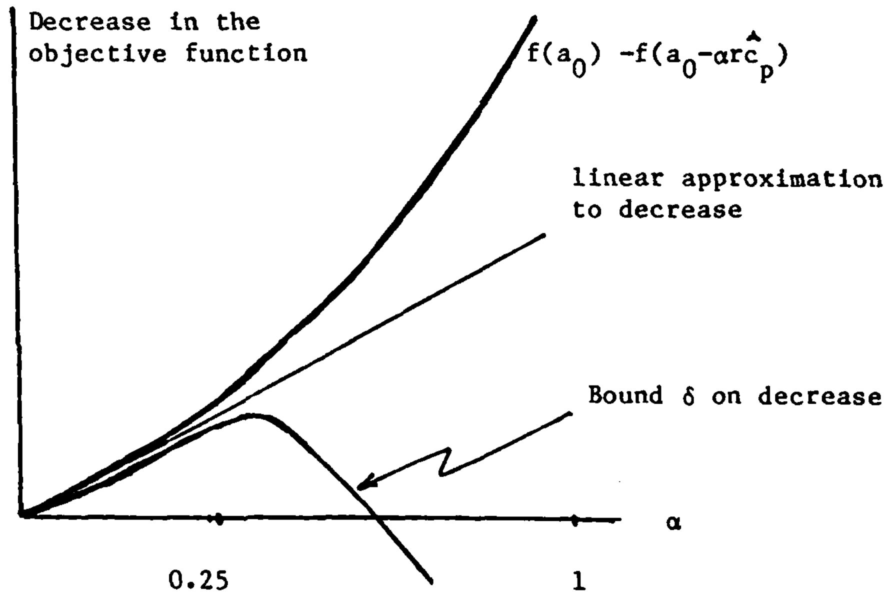

Figure 1. Bound δ given by (1)

Note that even though δ gives a good approximating bound for the decrease in the objective function for $\alpha < 1/4$, the bound for $\alpha = 1/4$ is valid for all $\alpha > 1/4$. Hence, the order of complexity is not adversely affected.

A second logical approach is to use the maximum value of α but to

proceed along the negative gradient direction of the objective function in TP. It seems probable that this will yield faster convergence even though it is not guaranteed to do so.

EXAMPLE

To illustrate the points made above we solved a simple problem proposed by Charnes et al. [2] using each of the above approaches and with different values of α.

Here, the original problem is

P: minimize x_2 s.t. $x_1 + x_2 + x_3 = 1$ $x_i > 0$ $i = 1,2,3$ and under a projective transformation T_a which maps an interior point $a = (a_1, a_2, a_3)$ of the feasible region of P to the center of the simplex S $= \{(y_1, y_2, y_3): y_1 + y_2 + y_3 = 1,\ y_i > 0 \text{ for } i = 1,2,3\}$ we get

$$\text{TP:} \quad \text{minimize } \frac{a_2 y_2}{a_1 y_1 + a_2 y_2 + a_3 y_3} \quad \text{s.t. } y \varepsilon S$$

Alternative 1: Proceed along the projected negative gradient $-\hat{c}_p$ of AP:

Following Karmarkar's notation, we have $B = [1,1,1]$ so that $c_p = [I - B^t(BB^t)^{-1}B]$ DC. Letting $\hat{c}_p = c_p / \|c_p\|$, and noting that the current point is $b = T_a(a) = (1/3, 1/3, 1/3)^t$, we have the new point

$$y = b - \alpha r \hat{c}_p = (\tfrac{1}{3} + \tfrac{\alpha}{6},\ 1/3 - \tfrac{\alpha}{3},\ \tfrac{1}{3} + \tfrac{\alpha}{6})^t$$

where $0 < \alpha < 1$ and $r = 1/\sqrt{n(n-1)} = 1/\sqrt{6}$. In the original space this point is

$$x = \frac{Dy}{e^t Dy} = \begin{cases} (\frac{a_1}{3} - \frac{\alpha}{6} a_1)/\Delta \\ (\frac{a_2}{3} - \frac{\alpha}{3} a_2)/\Delta \\ (\frac{a_3}{3} - \frac{\alpha}{6} a_3)/\Delta \end{cases}$$

where $\Delta = e^t Dy = \frac{1}{3} + \frac{\alpha}{6}(a_1 - 2a_2 + a_3)$

Alternative 2: Proceed along the projected negative gradient $-\hat{d}_g$ of TP: Here we have the projected gradient

$$d_g = [I - B^t(BB^t)^{-1}B]\nabla f(b) = \begin{cases} -3a_1a_2 \\ 3a_2(a_1+a_3) \\ -3a_2a_3 \end{cases}$$

where again the current point $b = (1/3,1/3,1/3)^t$: Letting $\hat{d}_g = d_g/\|d_g\|$, the new point is

$$y = b - \alpha r\hat{d}_g = \begin{cases} \frac{1}{3} + 3\alpha r a_1a_2/\|d_g\| \\ \frac{1}{3} + 3\alpha r a_2(a_1+a_3)/\|d_g\| \\ 1/3 + 3\alpha r a_2a_3/\|d_g\| \end{cases}$$

The corresponding point in the original space is then

$$x = \frac{Dy}{e^t Dy} = \begin{cases} (\frac{a_1}{3} + 3\alpha r a_1 a_2/\|d_g\|)/\bar{\Delta} \\ (\frac{a_2}{3} + 3\alpha r a_2(a_1+a_3)/\|d_g\|/\bar{\Delta} \\ (\frac{a_3}{3} + 3\alpha r a_2 a_3/\|d_g\|)/\bar{\Delta} \end{cases}$$

where $\bar{\Delta} = e^t Dy = \frac{1}{3} + (3\alpha r a_2/\|d_g\|)[a_1^2 - a_2(a_1+a_3) + a_3^2]$. The computational results for the two alternatives are shown in Tables 1 and 2 for $\alpha = 0.1$, 0.25, 0.50, 0.80, and 0.90. The entries in the tables are the objective function values. The starting point in each case is $(0.10, 0.30, 0.60)^t$.

This simple example strongly supports the use of α value of the order of 0.90. The results using the direction $-\hat{d}_g$ is comparable with that obtained using $-\hat{c}_p$. Clearly further computational testing is needed before any firm conclusions can be drawn. The order of complexity under alternative 2 also needs to be investigated.

References

1. Karmarkar, N., "A New Polynomial-Times Algorithm of Linear Programming," Technical Report, AT&T Bell Laboratories, New Jersey, 1984.

2. Charnes, A., T. Song, and M. Wolfe, "An Explicit Solution Sequence and Convergence of Karmarkar's Algorith," Research Report CCS 501, Center for Cybernetic Studies, The University of Texas, 1984.

Table 1.

Iteration #	= 0.10		= 0.25	
	Alternative 1	Alternative 2	Alternative 1	Alternative 2
1	.2686567164	.2666316506	.2222222222	.2190905811
6	.1452687100	.1329111912	.0362606232	.0284795456
11	.0729011799	.0594067529	.0049302946	.0030803808
16	.0351038815	.0253844452	.0006520483	.0003309294
21	.0165535577	.0106779681	.0000859151	.0000355966
26	.0077274757	.0044705145		
31	.0035901348	.0018695841		
36	.0016642379	.0007818099		
41	.0007706728	.0003269792		
46	.0003567105	.0001367727		
51	.0001650689	.0000572162		
56	.0000763783	.0000239366		
61	.0000353390	.0000100143		

Table 2.

Iteration #	= 0.50		= 0.80		= 0.90	
	Alternative 1	Alternative 2	Alternative 1	Alternative 2	Alternative 1	Alternative 2
1	.0425531915	.0425452529	.0576923077	.0668291352	.0287081340	.0423400480
2	.0174672489	.0167066197	.0086705202	.0106232594	.0020342431	.0040101851
3	.0070609003	.0063472042	.0012479201	.0016625524	.0001405588	.0003854873
4	.0028363765	.0023815758	.0001784652	.0002622838	.0000096950	.0000374621
5	.0011364847	.0008895885	.0000254989	.0000415232		
6	.0004549041	.0003317606	.0000036428	.0000065805		
7	.0001820113	.0001236585				
8	.0000728125	.0000460834				
9	.0000291263	.0000171728				
10	.0000116507	.0000063992				

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